10-2 day 2 Vector Calculus

Learning Objectives:

I can model motion using a vector equation

I can find the velocity, speed, acceleration, and direction vector of a particle whose motion is described by a vector equation

I can find the displacement, total distance traveled, and position of a particle whose motion is described by a vector equation

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Velocity, Speed, Acceleration, and Direction of Motion

Suppose a particle moves along a smooth curve in the plane so that its position at any time t is (x(t), y(t)) where x and y are differentiable functions of t.

1. The particle's position vector is p(t)=<x(t), y(t)>
2. The particle's velocity vector is v(t) = <x'(t), y'(t)>
3. The particle's speed is the magnitude of the velocity vector $s(t) = |v(t)| = \sqrt{(x'(t))^2 + (y'(t))^2}$
4. The particle's acceleration vector is a(t)=<x''(t),y''(t)>
5. The particles direction of motion is a unit vector called the direction vector and is given by $d(t) = \dfrac{v(t)}{|v(t)|}$

Ex1. A particle's position is given by

$x(t) = e^t\cos(t) \qquad y(t) = \sin^2(t)$

a.) Find the position vector

b.) Find the velocity vector

c.) Find the acceleration vector

d.) Find the position, velocity, and acceleration of the particle at time $t = \frac{\pi}{6}$

$x = e^t \cos t \qquad y = \sin t$

a.) Find the position vector

$$\boxed{r(t) = \langle e^t \cos t,\ \sin^2 t \rangle}$$

b.) Find the velocity vector

$$\boxed{v(t) = \langle e^t \cos t - e^t \sin t,\ 2 \sin t \cos t \rangle}$$

c.) Find the acceleration vector

$$a(t) = \langle e^t \cos t - e^t \sin t - (e^t \sin t + e^t \cos t),\ 2\cos t \cos t + 2 \sin t \cdot -\sin t \rangle$$

$$= \langle \cancel{e^t \cos t} - e^t \sin t - e^t \sin t \cancel{- e^t \cos t},\ 2\cos^2 t - 2 \sin^2 t \rangle$$

$$= \langle -2e^t \sin t,\ 2(\cos^2 t - \sin^2 t) \rangle$$

$$\boxed{a(t) = \langle -2e^t \sin t,\ 2 \cos 2t \rangle}$$

d.) Find the position, velocity, and acceleration of the particle at time $t = \pi/6$.

$$r(t) = \langle e^{\pi/6} \cos \tfrac{\pi}{6},\ \sin^2 \tfrac{\pi}{6} \rangle$$

$$= \langle e^{\pi/6} \tfrac{\sqrt{3}}{2},\ (1/2)^2 \rangle = \boxed{\langle \tfrac{\sqrt{3}}{2} e^{\pi/6},\ 1/4 \rangle}$$

$$v(t) = \langle e^{\pi/6} \cos \pi/6 - e^{\pi/6} \sin \pi/6,\ 2 \sin \pi/6 \cos \pi/6 \rangle$$

$$= \langle e^{\pi/6} \tfrac{\sqrt{3}}{2} - e^{\pi/6}\, 1/2,\ \cancel{2}\, \tfrac{\sqrt{3}}{\cancel{2}} \cdot 1/2 \rangle$$

$$= \boxed{\langle \tfrac{\sqrt{3}}{2} e^{\pi/6} - 1/2\, e^{\pi/6},\ \tfrac{\sqrt{3}}{2} \rangle}$$

$$a(t) = \langle -2e^{\pi/6} \sin \pi/6,\ 2 \cos \pi/3 \rangle$$

$$= \langle -\cancel{2} e^{\pi/6}\, 1/\cancel{2},\ \cancel{2}\, \tfrac{1}{\cancel{2}} \rangle = \boxed{\langle -e^{\pi/6},\ 1 \rangle}$$

e.) Find the speed at time t= $\frac{\pi}{6}$

f.) Find the direction vector at time t= $\frac{\pi}{6}$

e.) Find the speed at time $t = \pi/6$

$$S = |\vec{v}| = \sqrt{\left(\frac{\sqrt{3}}{2}e^{\pi/6} - \frac{1}{2}e^{\pi/6}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \sqrt{\left(\frac{\sqrt{3}}{2}e^{\pi/6} - \frac{1}{2}e^{\pi/6}\right)^2 + \frac{3}{4}}$$

$$\approx 1.064$$

f.) Find the direction vector at time $t = \pi/6$

$$\frac{\vec{v}}{|\vec{v}|} = \frac{\left\langle \frac{\sqrt{3}}{2}e^{\pi/6} - \frac{1}{2}e^{\pi/6}, \frac{\sqrt{3}}{2} \right\rangle}{1.064}$$

$$\approx \langle .581, .814 \rangle$$

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Displacement and Total Distance Traveled

Suppose a particle moves along a smooth curve in the plane so that its position at any time t so that its velocity is given by $v(t) = \langle v_x(t), v_y(t)\rangle$.

1. The displacement of the object from time t=a to time t=b is given by the vector: $\left\langle \int_a^b v_x(t)\,dt, \int_a^b v_y(t)\,dt \right\rangle$

2. The total distance traveled from time t=a to time t=b is

$$\int_a^b |v(t)|\,dt = \int_a^b \sqrt{(v_x(t))^2 + (v_y(t))^2}\,dt$$

$$\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

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Ex2. A particle's velocity is given by

$$v(t) = \left\langle \frac{1}{t}, t^2 \right\rangle$$

a.) Find the displacement of the object from time t=1 to time t=5.

b.) Find the total distance traveled from time t=1 to time t=5.

c.) If the particle was at position (-2,3) at time t=1, what is the position of the particle at time t=5?

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a.) Find the displacement of the object from $t=1$ to time $t=5$

$$= \left\langle \int_1^5 \frac{1}{t}\,dt,\ \int_1^5 t^2\,dt \right\rangle$$

$$= \left\langle \ln|t| \Big|_1^5,\ \tfrac{1}{3}t^3 \Big|_1^5 \right\rangle$$

$$= \left\langle \ln 5 - \ln 1,\ \tfrac{1}{3}(5)^3 - \tfrac{1}{3}(1)^3 \right\rangle$$

$$= \left\langle \ln 5,\ \tfrac{125}{3} - \tfrac{1}{3} \right\rangle = \left\langle \ln 5,\ \tfrac{124}{3} \right\rangle$$

b.) Find the total distance traveled from time $t=1$ to time $t=5$

$$= \int_1^5 \sqrt{\left(\tfrac{1}{t}\right)^2 + \left(t^2\right)^2}\,dt$$

$$= \int_1^5 \sqrt{\tfrac{1}{t^2} + t^4}\,dt \approx 41.488$$

c.) If the particle was at position $(-2, 3)$ at time $t=1$, what is the position of the particle at time $t=5$.

$$(-2,3) + \left\langle \ln 5, \tfrac{124}{3} \right\rangle = \left(-2 + \ln 5,\ 3 + \tfrac{124}{3}\right)$$

$$\approx \boxed{(-.391,\ 44\tfrac{1}{3})}$$

Homework

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